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Integrability criteria for differential-difference systems: a comparison of singularity confinement and low-growth requirements

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Abstract. A new approach for the study of the integrability of differential-difference systems is introduced. Given a differential-difference system (of such a form that it can be iterated without necessitating the integration of a differential equation at any step) a two-stage strategy is used. First, the singularity confinement necessary integrability criterion is used in order to limit the possible choices. Once the system is sufficiently reduced, the (much stronger) requirement of nonexponential growth of the degree of the iterates of some initial condition is implemented. This method turns out to be powerful and practical. The investigation of a given class of differential-difference systems has resulted in some well known integrable systems but also to two promising integrability candidates (one of which is reduced to a known integrable case).

Integrable discrete systems have received much attention over the past few years resulting in a much better understanding of their properties. In particular, discrete integrability criteria have been proposed [1–3] and their judicious use could turn them into efficient integrability detectors. The extension of these purely discrete methods to systems which are both discrete and continuous is not trivial. The complication comes from the fact that these systems must also satisfy the integrability requirements for their continuous part. Thus the right approach to the study of the integrability of differential-difference systems should be based on a blending of the continuous integrability methods and their discrete counterparts. In [4, 5] we have presented a first approach to the study of the integrability of discrete–continuous systems. It was based on the combination of the Painlevé singularity analysis, for the continuous part, and the singularity confinement approach for the discrete part. The continuous singularity analysis [6] consists in the requirement of an absence of critical movable singularities in the solutions of a nonlinear differential equation (Painlevé property). The singularity confinement [1] is based on the observation that a singularity, that appears spontaneously in an integrable discrete system, disappears after some iterations. The discovery of this property led us to propose singularity confinement as a necessary integrability criterion. Its use, essentially as a tool for the de-autonomization of integrable discrete systems, made possible, among others, the discovery of discrete and q-discrete Painlevé equations. In a recent paper Hietarinta and Viallet [3] presented examples of mappings which although satisfying the singularity confinement requirement were not integrable. Thus singularity confinement is not a sufficient criterion (a fact already noted in [7]). What these authors proposed, in the particular case of birational mappings, is a criterion based on the ideas of Arnold [8] and Veselov [9], on the growth of the degrees of the iterates of some initial data under the action of the mapping. The main argument is that a generic, nonintegrable, mapping has an exponential degree growth, while integrability is associated with low growth, typically polynomial. In [10] we have examined the implications of this approach on discrete Painlevé equations. We were able to show that the conclusions based on singularity confinement were in total agreement with those of the nonexponential growth requirement. Thus the use of singularity confinement for the de-autonomization of a given integrable discrete system is perfectly justified (and considerably simpler than the study of degree growth).

This paper is devoted to the study of the integrability of differential-difference systems based on the low-growth requirements. We shall first apply this method to well known integrable systems which were already examined in [5]. We shall then look for generalizations of these systems using a mixed approach. Starting from a given functional form we first apply singularity confinement in order to reduce the number of free parameters and, once the problem becomes tractable, we use the low-growth criterion in order to obtain the final answer. It turns out that in all the cases examined here the result of the application of this stringent criterion coincides with that of singularity confinement (but this is perhaps due to the particular family of systems we study).

The equations we will examine are of the general form $u_{n+1} = F(u_{n-1}, u_n, u'_n, n, t)$ with F homographic in u_{n-1}, rational in u_n, u'_n and analytic in n, t and where the prime denotes the derivative with respect to t. Given an equation of this form, we iterate initial conditions in homogeneous coordinates $u_0 = p$, $u_1 = q/r$, where p, q, r are functions of t. We assign to p (and t) the degree zero and we compute the degree d_n of homogeneity in q and r of the numerator and denominator of u_n at every iteration. A different choice of u_0 could have been possible but it turns out that the present choice of zero-degree u_0 considerably simplifies the calculations.

We shall start with two well known integrable systems. The first is the Kac–Moerbeke equation [11], also known as the Lotka–Volterra or semi-discrete KdV equation:

$$u_{n+1} = u_{n-1} + \frac{u'_n}{u_n}. \tag{1}$$

Let us explain how the degree growth is computed. We start from $u_0 = p$, $u_1 = q/r$ and compute the first few iterates of (1). We thus obtain

$$u_2 = \frac{pqr - q'r - qr'}{qr}$$
$$u_3 = \frac{q^2(pqr - q'r - qr' + r'^2 - rr'') + r^2(p'q^2 + qq'' - q'^2)}{(pqr - q'r - qr')qr}$$

and so on. Since p and t are of degree zero and q and r of degree one we find that the homogeneity degrees of the numerator and denominator of u_2 and u_3 are respectively $d_2 = 2$ and $d_3 = 4$. Computing the degree of the successive iterates we find $d_n = 0, 1, 2, 4, 7, 11, 16, 22, \ldots$, i.e. given by $d_n = (n^2 - n + 2)/2$ for $n > 0$. (A remark seems necessary at this point concerning the closed-form expression of d_n. In order to obtain it we start by computing a sufficient number of d_n and heuristically establish this expression. We then compute the next few degrees and compare their values with the ones obtained analytically.) The fact that the degree growth is polynomial is not astonishing given that the Kac–Moerbeke system is integrable. The second system we shall examine is the semi-discrete mKdV equation [12]:

$$u_{n+1} = u_{n-1} + \frac{u_n'}{u_n^2 - 1}. \tag{2}$$

Again we find a polynomial growth $d_n = 0, 1, 2, 5, 8, 13, 18, 25, \ldots$. We have indeed $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2 + 2m + 1$. Again, nonexponential growth is expected since the integrability of (2) is well established. Several more differential-difference equations have been treated along the same lines resulting in every case in a polynomial growth of the degree of the iterates.

Once our approach has passed this basic test it is natural to ask how we can generalize equations (1) and (2). In order to keep this search for generalizations manageable we shall limit ourselves to equations of the form:

$$u_{n+1} = u_{n-1} + \frac{\alpha u_n' + \beta u_n^2 + \gamma u_n + \delta}{\kappa u_n' + \zeta u_n^2 + \eta u_n + \theta} \tag{3}$$

where $\alpha, \ldots, \theta, \kappa$ are functions of n and t. Our approach will be based on a dual singularity confinement/low-growth requirement strategy. We shall start by reducing the possible integrable forms of (3) using the necessary criterion of singularity confinement and then analyse the reduced form through the study of the degree growth. We start by supposing that $\kappa \neq 0$, (we take $\kappa = 1$) in which case by translation we can put $\alpha = 0$. Let us assume that u_n is such that $u_n' + \zeta u_n^2 + \eta u_n + \theta$ has a simple zero for some $t = t_0$. In this case u_{n+1} will have a simple pole, $u_{n+1} \propto 1/(t - t_0)$. This singularity will propagate indefinitely, i.e. u_{n+3}, u_{n+5} etc will also have poles unless the following conditions are fulfilled: $\beta = \gamma = \zeta = 0$, and $\eta_{n+1} = \eta_{n-1}$, $\theta_{n+1} = \theta_{n-1}$, $\delta_{n+1} - 2\delta_n + \delta_{n-1} = 0$. Thus δ is linear in n while η and θ are n-independent with even/odd dependence, which means that we have $\eta_e(t)$, $\theta_e(t)$ for even n and different $\eta_o(t)$, $\theta_o(t)$ for odd n. Introducing $u = \xi_{e,o} v$, where $\xi_{e,o} = \exp(-\int \eta_{e,o}\, \mathrm{d}t)$ we can transform the equation to

$$\xi_e \xi_o (v_{n+1} - v_{n-1}) = \frac{\delta_n}{v_n' + \theta_{e,o}/\xi_{e,o}}. \tag{4}$$

The factor $\xi_e \xi_o$ can be absorbed in δ and a translation of v, by $\int \theta_{e,o}/\xi_{e,o}\, \mathrm{d}t$, allows us to put θ in the denominator of (4) to zero. Thus we arrive finally at the equation

$$v_{n+1} - v_{n-1} = \frac{\lambda(t) n + \mu(t)}{v_n'} \tag{5}$$

where, moreover, it is possible to take $\lambda = 1$ through a suitable redefinition of time. This equation, as explained above, is a candidate for integrability. Once this reduced form is obtained through singularity confinement we can apply the nonexponential growth criterion. We start by considering the equation $v_{n+1} - v_{n-1} = a(n, t)/v_n'$ where a is *a priori* an arbitrary function of n and t. We compute, as in the case of systems (1) and (3), the degree growth starting from $v_0 = p$ and $v_1 = q/r$ and obtain the exponentially growing sequence $d_n = 0, 1, 2, 4, 8, 16, \ldots$, i.e. $d_n = 2^{n-1}$ for $n > 0$. Next we ask how it is possible to curb this growth and it turns out that we can, for $n = 4$, obtain a condition for the degree to be six rather than eight. This condition is $a_{n+1} - 2a_n + a_{n-1} = 0$, i.e. a must be a linear function of n, in perfect agreement with the singularity confinement criterion. Implementing this constraint we can now compute the degree growth for equation (5). We now obtain the sequence $d_n = 0, 1, 2, 4, 6, 9, 12, 16, \ldots$, i.e. $d_{2m-1} = m^2$ and $d_{2m} = m(m+1)$, which are precisely the same values as the ones obtained when a is a constant (in both n and t). Thus the low-growth requirement criterion confirms the possibly integrable character of (5). Although this is not a *proof* of its integrability, the fact that this new criterion is satisfied strengthens the argument in favour of integrability. We shall return to this equation and show that it can be transformed to a known integrable system. For the time being we compute its continuous limit. Equation (5) is another differential-difference form of the potential KdV equation. Introducing the continuous variables $x = \epsilon(n+t)$, $s = \epsilon^3 t$ and taking $v(n,t) = n - t + \epsilon w(x,s)$, $a(n,t) = 2(-1 + \epsilon^4(b'(s)x + c(s))$ we find at the limit $\epsilon \to 0$ the equation

$$w_s + w_x^2 - \tfrac{1}{6} w_{xxx} = b'(s)x + c(s). \tag{6}$$

This is indeed a potential form of KdV. Differentiating once with respect to x we obtain for the quantity $W = w_x - b(s)$ the equation

$$W_s + 2WW_x - \tfrac{1}{6} W_{xxx} = -2b(s)W_x. \tag{7}$$

Introducing the new variables $T = s$ and $X = x - 2\int b(s)\,\mathrm{d}s$ we finally find

$$W_T + 2WW_X - \tfrac{1}{6} W_{XXX} = 0 \tag{8}$$

i.e. the KdV equation.

Next we consider (3) in the particular case $\kappa = 0$. In this case and provided $\zeta \neq 0$ a translation allows us to take $\beta = 0$ and, without loss of generality, the equation can be rewritten:

$$u_{n+1} = u_{n-1} + \alpha \frac{u_n' + \gamma u_n + \delta}{(u_n - \rho)(u_n - \sigma)}. \tag{9}$$

The singularity confinement analysis of (9) can be easily performed. Two confined singularity patterns exist: $\{\rho, \infty, \sigma\}$ and $\{\sigma, \infty, \rho\}$. Implementing the conditions for their existence finally leads to the equation

$$u_{n+1} = u_{n-1} + \left(\frac{\alpha}{\rho}\right) \frac{\rho u_n' - \rho' u_n}{u_n^2 - \rho^2} \tag{10}$$

where $\alpha = \lambda(t)n + \mu(t)$ and $\rho = \phi_{e,o}\alpha$. Introducing the new variable $v = u/\rho$ we can transform equation (10) to

$$\alpha_{n+1} v_{n+1} - \alpha_{n-1} v_{n-1} = \frac{v_n'}{v_n^2 - 1} \tag{11}$$

where we have redefined time $\mathrm{d}t \to \phi_o \phi_e\,\mathrm{d}T$. Thus the final form of this equation which should be a candidate for integrability according to the singularity confinement criterion is

$$(n + 1 + \nu(t))v_{n+1} - (n - 1 + \nu(t))v_{n-1} = \frac{v_n'}{v_n^2 - 1} \tag{12}$$

where the time has to be redefined so as to absorb the λ factor, $\nu = \mu/\lambda$ is an arbitrary function of the new time and the prime denotes derivation with respect to this new independent variable. At this stage we resort again to the study of the growth of the degree of the iterates starting from equation (11) where we assume a generic α. The degrees obtained are $d_n = 0, 1, 2, 5, 12, 29, \ldots$, obeying the recursion relation $d_{n+1} = 2d_n + d_{n-1}$ leading to an exponential growth with asymptotic ratio $1 + \sqrt{2}$. The condition for d_4 to be less than 12 is $\alpha_{n+1} - 2\alpha_n + \alpha_{n-1} = 0$, i.e. α linear in n, exactly the same condition as the one resulting from singularity confinement. Implementing this condition results in a polynomial growth identical to that of equation (2), which is its autonomous (in both n and t) counterpart. Thus equation (12) should be integrable, and indeed it is. As was shown by Cherdantsev and Yamilov [13], (see also [14]), this equation is the master symmetry of the semi-discrete mKdV.

Finally, we examine the case $\kappa = \zeta = 0$. If η were also zero, the right-hand side would be polynomial and no simplifications could ever curb the exponential growth of the iterates. Thus without loss of generality we can take $\eta = 1$ and translation allows us to put $\theta = 0$.

The implementation of the singularity confinement criterion, when u has a simple zero, leads to the following constraints: $\delta = 0$, $\beta_n = \alpha_n(\alpha_{n+1}^{-1} - \alpha_{n-1}^{-1})$ and, introducing the auxiliary quantity $a_n = \alpha_n\alpha_{n+1}$, we find that a must obey the recurrence $a_{n+1} - 2a_n + a_{n-1} = 0$, i.e. a is linear in n. Before proceeding further we transform the equation by introducing the change of variables: $u_n = v_n\alpha_{n+1}\alpha_n\alpha_{n-1}$. We thus find

$$a_{n+1}v_{n+1} - a_{n-2}v_{n-1} = \frac{v_n'}{v_n} + (a_{n-1} - a_n)v_n + \sigma_n. \tag{13}$$

Further implementing the singularity confinement criterion we find that σ must be n-independent in which case a t-dependent gauge of v can put it to zero. The final form of our equation is

$$a_{n+1}v_{n+1} - a_{n-2}v_{n-1} = \frac{v_n'}{v_n} + (a_{n-1} - a_n)v_n. \tag{14}$$

Next we implement the low-growth criterion on equation (14) without making any assumption on a. We find the degree sequence $d_n = 0, 1, 2, 4, 8, 16, \ldots$, i.e. $d_n = 2^{n-1}$ for $n > 0$. In order to limit this exponential growth we require that the degree d_4 be seven instead of eight. We readily find the condition $a_{n+1} - 2a_n + a_{n-1} = 0$, i.e. exactly the one expected by singularity confinement. Iterating further we find that provided a_n is linear in n, the degree growth is identical to that of equation (1). This is quite natural since equation (14) is the master symmetry of the Kac–Moerbeke equation and thus constitutes its integrable nonautonomous extension [13].

Having obtained equation (14) we can now return to equation (5) and show how it can be integrated. Starting from (5) we introduce $w_n = v_{n+1} - v_{n-1}$. We then have for w the equation

$$\frac{w_n'}{w_n} = \frac{a_{n+1}}{w_n w_{n+1}} - \frac{a_{n-1}}{w_n w_{n-1}}. \tag{15}$$

Next we introduce the variable $u_n = -1/w_n w_{n-1}$ and using (15) we recover equation (14) exactly; the nonautonomous extension to Kac–Moerbeke we have just obtained.

We turn now to a second class of equations. Their study is motivated by the results of [15] on differential-delay systems. We start from the general form

$$u_{n+1} = u_{n-1}\frac{\alpha u_n' + \beta u_n^2 + \gamma u_n + \delta}{\kappa u_n' + \zeta u_n^2 + \eta u_n + \theta}. \tag{16}$$

First we perform the singularity confinement analysis on (16). We shall not present all the details here but just give the final result:

$$u_{n+1} = u_{n-1}\frac{u_n' + (\gamma_{e,o}' - \lambda n - \mu)u_n}{u_n' + (\gamma_{e,o}' + \lambda n + \mu)u_n} \tag{17}$$

where $\gamma_{e,o}$ is a function of t with even/odd dependence and λ, μ are also functions of t. By introducing the gauge $u_e = v_e e^{\gamma_e}$ and $u_o = v_o e^{\gamma_o}$ for u of even and odd indices we can put γ to zero in (17). Thus the final form of (17), which would be a candidate for integrability according to singularity confinement is

$$v_{n+1} = v_{n-1}\frac{v_n' - (\lambda n + \mu)v_n}{v_n' + (\lambda n + \mu)v_n}. \tag{18}$$

We now turn to a degree-growth analysis and start from an equation $v_{n+1}/v_{n-1} = (v_n' - a(n,t)v_n)/(v_n' + a(n,t)v_n)$. For a generic a we obtain a degree sequence $d_n = 0, 1, 2, 5, 12, 29, \ldots$, i.e. again an exponential growth with asymptotic ratio $1+\sqrt{2}$. Requiring a nonexponential growth we obtain the constraint for a: $a_{n+1} - 2a_n + a_{n-1} = 0$, i.e.

an a linear in n, just as predicted by confinement. In this case the degree sequence is $d_n = 0, 1, 2, 5, 8, 13, 18, 25, \ldots$, i.e. $d_{2m} = 2m^2$ and $d_{2m+1} = 2m^2 + 2m + 1$, a polynomial growth, precisely the same values as the case of constant a. Thus we expect system (18), based on both singularity confinement and low-growth criteria, to be integrable. We cannot offer a further proof for its integrability but we shall show that at the continuous limit it goes over to an integrable equation. Indeed, we introduce the continuous variable $x = \epsilon n$ and put $a = 1/\epsilon + \epsilon(b'(t)x + c(t))$ while transforming v through $v = \mathrm{e}^w$. We find thus the equation

$$w_t = w_x + \epsilon^2(b'(t)x + c(t))w_x + \frac{\epsilon^2}{6}(w_{xxx} - 2w_x^3). \tag{19}$$

Next we perform a Galilean transformation $\partial_t \to \partial_x + \epsilon^2\partial_t$ and reduce (19) to

$$w_t = (b'(t)x + c(t))w_x + \tfrac{1}{6}(w_{xxx} - 2w_x^3). \tag{20}$$

Finally, introducing the new variables $T = \int \mathrm{e}^{3b(t)}\,\mathrm{d}t$ and $X = x\mathrm{e}^{b(t)} + \int c(t)\mathrm{e}^{b(t)}\,\mathrm{d}t$ we obtain

$$w_T = w_{XXX} - 2w_X^3 \tag{21}$$

which is just the potential modified-KdV.

In this paper we have examined some selected cases of differential-difference systems, investigating their integrability. Our method was a new approach which combines the singularity confinement (which is a well-established necessary integrability criterion) and the low-growth requirement which provides a more stringent test at the price of considerably heavier computations. Thus singularity confinement is used first in order to limit the freedom of the initial ansatz and the low-growth criterion is subsequently implemented in order to eliminate the nongenuine integrability candidates. We expect this method to become a very powerful tool for the study of discrete integrability.

One interesting result is that in all cases investigated here the constraints provided by the confinement requirement were sufficient and the implementation of the low-growth criterion did not invalidate the conclusions. This result closely follows our conclusions in the case of discrete Painlevé equations and may be due to the structure of the particular class of equations we are studying.

The approach presented above is suitable for differential-difference systems where one does not have to integrate a differential equation along the way. One can then easily assign weights to the various variables, compute the homogeneous degree and estimate the rate of growth. However, there exist systems in which one does have to integrate a differential equation at each iteration step. The differential-difference systems of the form $F(u'_{n+1}, u_{n+1}, u'_n, u_n, n, t) = 0$ we examined in [5, 15] are of this type. The requirement for the Painlevé property of the differential equation we have at each step would, in this particular case, mean that the equation obeyed by each u_n must be a Riccati equation. Although the degree-growth criterion cannot be applied to such systems we still can study them in the framework of singularity analysis. An equation like the one considered in this paragraph can be viewed perfectly as an infinite system of coupled differential equations. Thus a movable singularity (which must be noncritical for the Painlevé property to be satisfied) appearing at some iteration step will be present in the coefficients of the terms of the differential equations corresponding to the subsequent iterations. Thus the systems of this type can be studied with methods perfectly appropriate for purely differential systems.

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References

[1] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[2] Conte R and Musette M 1996 *Phys. Lett.* A **223** 439
[3] Hietarinta J and Viallet C-M 1998 *Phys. Rev. Lett.* **81** 325
[4] Ramani A, Grammaticos B and Tamizhmani K M 1992 *J. Phys. A: Math. Gen.* **25** L883
[5] Ramani A, Grammaticos B and Tamizhmani K M 1993 *J. Phys. A: Math. Gen.* **26** L53
[6] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333
[7] Grammaticos B, Ramani A and Tamizhmani K M 1994 *J. Phys. A: Math. Gen.* **27** 559
[8] Arnold V I 1990 *Bol. Soc. Bras. Mat.* **21** 1
[9] Veselov A P 1992 *Commun. Math. Phys.* **145** 181
[10] Ohta Y, Tamizhmani K M, Grammaticos B and Ramani A 1999 Singularity confinement and algebraic entropy: the case of the discrete Painlevé equations *Preprint*
[11] Kac M and van Moerbeke P 1975 *Adv. Math.* **16** 160
[12] Ablowitz M J and Ladik J F 1977 *Stud. Appl. Math.* **57** 1
[13] Cherdantsev I and Yamilov R 1995 *Physica* D **87** 140
[14] Grammaticos B, Ramani A and Tamizhmani K M De-autonomization of differential-difference equations and discrete Painlevé equations *Chaos, Solitons and Fractals* to appear
[15] Grammaticos B, Ramani A and Moreira I 1993 *Physica* A **196** 574